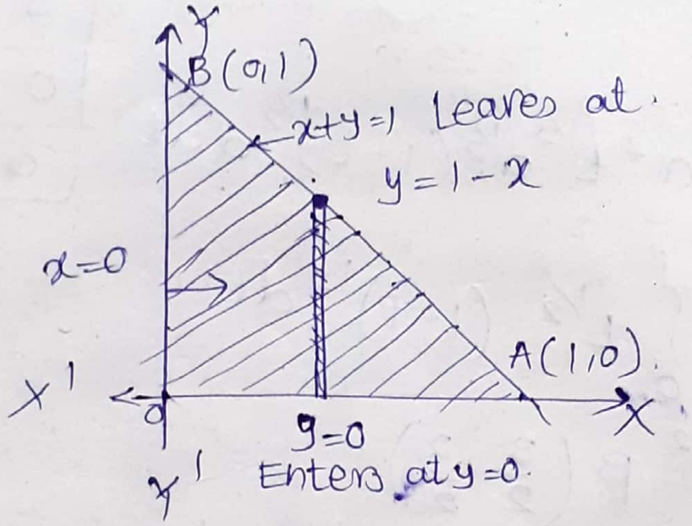
Multiple Integral.

(i) Evaluate $\iint_R \sqrt{xy(1-x-y)}\, dx\, dy$ over the region bounded by $x=0$, $y=0$ and $x+y=1$

Soln:- Let $I = \iint_R \sqrt{xy(1-x-y)}\, dx\, dy$ —— (1)

where $R(x=0,\ y=0,\ x+y=1)$



To find limits for x & y draw a vertical strip (parallel to y-axis) in the region of integration over which y varies from $y=0$ to $y=1-x$.

Now, moving the strip from $x=0$ to $x=1$ we get completed shaded region of Integration.

from eqn (1),

$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \sqrt{x}\sqrt{y(1-x-y)}\, dx\, dy$$

(w.r.t. y first keeping x as const)

$$= \int_{x=0}^{1} \sqrt{x} \left[\int_{y=0}^{1-x} \sqrt{y(1-x-y)}\, dy \right] dx.$$

$$= \int_{x=0}^{1} \sqrt{x}\, [I_1]\, dx \quad \text{——— (2)}$$

where $I_1 = \int_0^{1-x} \sqrt{y(1-x-y)}\,dy$ If $1-x=a$

$$= \int_0^a \sqrt{y(a-y)}\,dy$$

$\because$ put $y = at$

$dy = a\,dt$

y	t
0	0
a	1

$$I_1 = \int_0^1 \sqrt{at(a-at)}\;a\,dt$$

$$= \int_0^1 a^{1/2} t^{1/2} a^{1/2} (1-t)^{1/2} a\,dt$$

$$= a^2 \int_0^1 t^{1/2} (1-t)^{1/2}\,dt$$

$$= a^2\, \beta\left(\frac{3}{2}, \frac{3}{2}\right)$$

$$= a^2 \frac{\sqrt{3/2}\cdot\sqrt{3/2}}{\sqrt{3}}$$

$$= a^2 \frac{\frac{1}{2}\sqrt{1/2}\cdot\frac{1}{2}\sqrt{1/2}}{2}$$

$$\boxed{I_1 = \frac{\pi a^2}{8}} \quad \text{--- (3)}$$

from eq^n (2) + (3) $I = \int_0^1 x^{1/2} \frac{\pi a^2}{8}\,dx.$

$$= \frac{\pi}{8}\int_0^1 x^{1/2}(1-x)^2\,dx$$

$$= \frac{\pi}{8}\,\beta\left(\frac{3}{2}, 3\right) = \frac{\pi}{8} \frac{\sqrt{3/2}\,\sqrt{3}}{\sqrt{9/2}}$$

$$= \frac{\pi}{8} \frac{\frac{1}{2}\sqrt{1/2}\cdot 2}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{1/2}} = \frac{2\pi}{105}$$

Tripl
Def^n
dt

Scanned by CamScanner

Triple Integration:-

Def^n:- Let $f(x,y,z)$ be a function which is continuous at every point of finite region (volume v) of three dimensional space. Divide the region v into n subregions of respective volumes $\delta v_1, \delta v_2, \dots \delta v_n$.

Let (x_r, y_r, z_r) be a point in the r^{th} subregion then sum

$$\iiint_V f(x,y,z)\,dv = \lim_{\substack{n\to\infty \\ \delta v_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r, z_r)\,\delta v_r \quad \text{--- (1)}$$

is called triple integration of $f(x,y,z)$ over the region v provided limit on R.H.S. of eq^n (1) exists.

(II) Evaluation of Triple integration:-

The triple integration.

$$I = \iiint_V f(x,y,z)\,dv.$$

$$= \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dx\,dy\,dz.$$

can be evaluated by successive integrals as follows:

$$I = \int_{x=a}^{x=b} \left(\int_{y=\phi_1(x)}^{y=\phi_2(x)} \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz \right] dy \right) dx \quad \text{--- (1)}.$$

Example:- Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$$

Sol^n:- Let

$$I = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \left[\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}} \right] dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\int_{z=0}^{m} \frac{dz}{\sqrt{m^2-z^2}} \right] dx\,dy \qquad \because \text{put } m = \sqrt{1-x^2-y^2} = \text{const}.$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{m}\right) \right]_0^m dx\,dy$$

$$= \int_0^1 \int_{y=0}^{\sqrt{1-x^2}} \left[\frac{\pi}{2}\right] dx\,dy$$

$$= \frac{\pi}{2}\int_0^1\left[\int_0^{\sqrt{1-x^2}} dy\right] dx$$

$$= \frac{\pi}{2}\int_0^1 (y)_0^{\sqrt{1-x^2}}\, dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1-x^2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$= \frac{\pi}{2}\left[\frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$I = \frac{\pi^2}{8}.$$

Example:- Evaluate $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\, dz\,dx\,dy$.

Soln:- Let

$$I = \int_0^1 \int_{y^2}^1 x\left[\int_{z=0}^{1-x} dz\right] dx\,dy$$

(w.r.t z first keeping x, y constant)

$$= \int_0^1 \int_{y^2}^1 x\left[(z)_0^{1-x}\right] dx\,dy$$

$$= \int_0^1 \int_{x=y^2}^1 x(1-x)\, dx\,dy$$

$$= \int_0^1 \left[\int_{x=y^2}^1 (x-x^2)\, dx\right] dy$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{y^2}^1 dy$$

$$= \int_0^1 \left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{y^4}{2}-\frac{y^6}{3}\right)\right] dy$$

$$= \int_0^1 \left(\frac{1}{6}-\frac{y^4}{2}+\frac{y^6}{3}\right) dy$$

$$= \left(\frac{1}{6}y-\frac{y^5}{10}+\frac{y^7}{21}\right)_0^1$$

$$= \frac{1}{6}-\frac{1}{10}+\frac{1}{21}$$

$$\boxed{I = \frac{4}{35}}$$

Transformation of triple integral is spherical polar, cylindrical polar co-ordinate if limits are not given:

(a) cartesian to spherical polar:-

put $x = r\sin\theta\cos\phi$.

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

$x^2+y^2+z^2 = r^2$

and $dx\,dy\,dz = |J|\,d\theta\,d\phi\,dr$

$$= \left|\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\right| d\theta\,d\phi\,dr.$$

$$= r^2\sin\theta\,d\theta\,d\phi\,dr.$$

$$I = \iiint_V F(x,y,z)\,dx\,dy\,dz.$$

$$= \iiint_V F(r,\theta,\phi)\,r^2\sin\theta\,d\theta\,d\phi\,dr.$$

* some standard limits:-

① For complete sphere $x^2+y^2+z^2 = a^2$.

r varies from $r=0$ to $r=a$.

θ varies from $\theta=0$ to $\theta=\pi$

ϕ varies from $\phi=0$ to $\phi=2\pi$

(ii) For hemisphere $x^2+y^2+z^2=a^2$

r varies from $r=0$ to $r=a$

θ varies from $\theta=0$ to $\theta=\frac{\pi}{2}$

ϕ varies from $\phi=0$ to $\phi=2\pi$

(iii) For ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

put $x=ar\sin\theta\cos\phi$

$y=br\sin\theta\sin\phi$

$z=cr\cos\theta$.

$dx\,dy\,dz = abc\,r^2\sin\theta\,dr\,d\theta\,d\phi$.

r varies from $r=0$ to $r=1$

θ varies from $\theta=0$ to $\theta=\pi$

ϕ varies from $\phi=0$ to $\phi=2\pi$

(b) cylindrical polar co-ordinates :-

put $x=\rho\cos\phi$.

$y=\rho\sin\phi$.

$z=z$.

$\therefore x^2+y^2=\rho^2$.

& $dx\,dy\,dz = |J|\,d\rho\,d\phi\,dz$

$$= \left|\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)}\right| d\rho\,d\phi\,dz.$$

$dx\,dy\,dz = \rho\,d\rho\,d\phi\,dz$.

(c) Dirichlet's Theorem :-

(1) For two variables x & y

$$\iint x^{a-1}y^{b-1}\,dx\,dy = \frac{\Gamma a\cdot\Gamma b}{\Gamma a+b+1} \quad \text{where } x+y\le 1$$

(2) For three variables x, y, z

$$\iiint x^{a-1}y^{b-1}z^{c-1}\,dx\,dy\,dz = \frac{\Gamma a\,\Gamma b\,\Gamma c}{\Gamma 1+a+b+c}$$

Q:- Evaluate $\iiint_V \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$ where V is annulus between the spheres $x^2+y^2+z^2=a^2$ & $x^2+y^2+z^2=b^2$ $(a>b>0)$

Sol:- Transforming the triple integral into spherical polar co-ordinates by putting.

$x = r\sin\theta\cos\phi$.

$y = r\sin\theta\sin\phi$.

$z = r\cos\phi$

$\therefore x^2+y^2+z^2 = r^2$ ——— (i)

$dx\,dy\,dz = r^2\sin\theta\, dr\, d\theta\, d\phi$.

$x^2+y^2+z^2 = a^2$ ——— (ii)

$\Rightarrow$ (i) & (ii) $\Rightarrow r = a$

& $x^2+y^2+z^2=b^2$

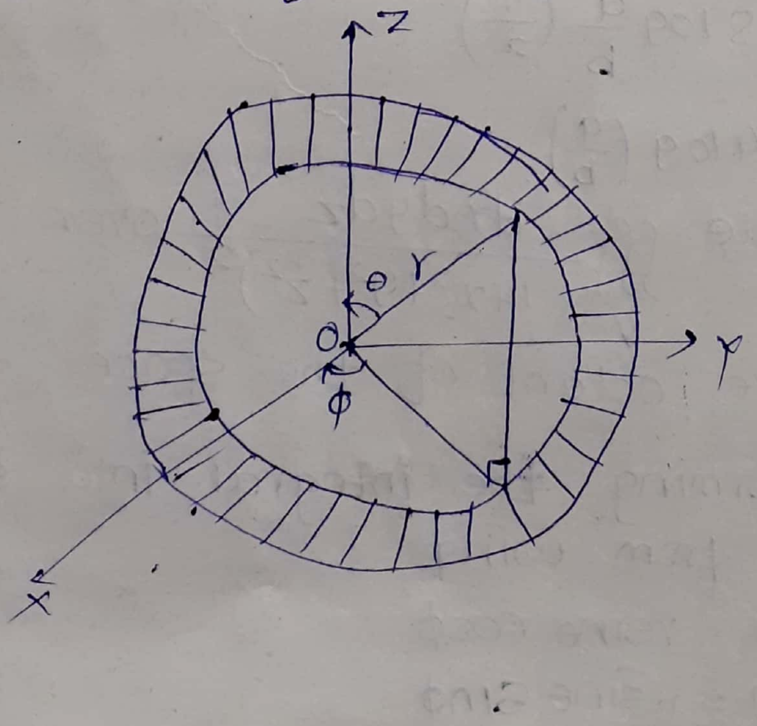
$\Rightarrow r = b$ $(b<r<a)$

For the positive octant

$\therefore$ r varies from $r=b$ to $r=a$.

θ varies from $\theta=0$ to $\theta = \cancel{\pi/2}\ 2\pi$

ϕ varies from $\phi=0$ to $\phi = \cancel{\pi/2}\ 2\pi$.



$$\therefore I = \iiint_V \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\int_{r=b}^{a} \frac{r^2\sin\theta\, d\theta\, d\phi\, dr}{(r^2)^{3/2}}$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\int_{r=b}^{a}\frac{1}{r}\,dr\right]\sin\theta\, d\theta\, d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}\left[\log r\right]_b^a \sin\theta\, d\theta\, d\phi$$

$$= 8\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2}(\log a - \log b)\sin\theta\, d\theta\, d\phi$$

$$= 8\log\frac{a}{b}\int_{\phi=0}^{\pi/2}\left[\int_{\theta=0}^{\pi/2}\sin\theta\, d\theta\right]d\phi$$

$$= 8\log\frac{a}{b}\int_{\phi=0}^{\pi/2}(-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= 8\log\frac{a}{b}\left(\frac{\pi}{2}\right)$$

$$= 4\log\left(\frac{a}{b}\right)$$

Que:- Evaluate $\iiint_V \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$ over the entire positive octant of the space.

Sol^n:- Transforming the integral into spherical polar from using

$x = r\sin\theta\cos\phi$.

$y = r\sin\theta\sin\phi$.

$x^2+y^2+z^2=r^2$

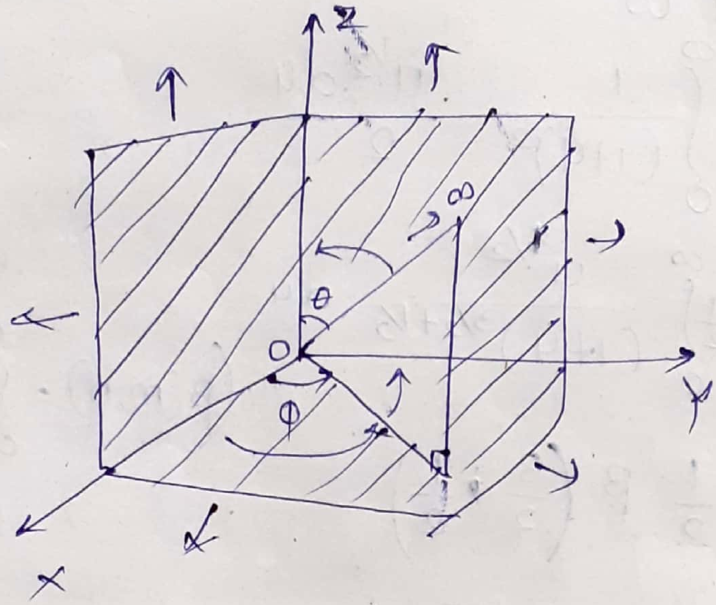
$\therefore dx\,dy\,dz = r^2 \sin\theta \, dr\, d\theta\, d\phi$.

over the positive octant

r varies from r to 0, to $r=\infty$.

θ varies from $\theta=0$ to $\theta=\pi/2$

ϕ varies from $\phi=0$ to $\phi=\pi/2$.



From the given integral

$$\therefore I = \iiint_V \frac{dx\,dy\,dz}{(1+x^2+y^2+z^2)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} \frac{r^2 \sin\theta \, d\theta\, d\phi\, dr}{(1+r^2)^2}$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \left[\int_{r=0}^{\infty} \frac{r^2 dr}{(1+r^2)^2} \right] d\theta\, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \sin\theta \, [I_1]\, d\theta\, d\phi \quad \text{——— (1)}$$

where $I_1 = \int_0^{\infty} \frac{r^2\, dr}{(1+r^2)^2}$

put $r^2 = u$

$\therefore 2r\,dr = du$

$r\,dr = \frac{du}{2}$

$r^2\,dr = \frac{\sqrt{u}}{2}\,du$

r	0	∞
u	0	∞

$$I_1 = \int_0^\infty \frac{1}{(1+u)^2} \cdot \frac{u^{1/2}\,du}{2}$$

$$= \frac{1}{2}\int_0^\infty \frac{u^{3/2}}{(1+u)^{3/2+1/2}}\,du \qquad \because \left(\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx\right)$$

$$= \frac{1}{2}\,\beta\left(\frac{3}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\,\frac{\Gamma 3/2\;\Gamma 1/2}{\Gamma 2}$$

$$= \frac{1}{2}\,\frac{\frac{1}{2}\Gamma 1/2\;\Gamma 1/2}{1}$$

$$= \frac{\pi}{4} \text{ ——— } (2)$$

$\therefore$ from eq^n (1)

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left(\frac{\pi}{4}\right) d\theta\,d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2}\left[\int_0^{\pi/2} \sin\theta\,d\theta\right] d\phi$$

$$= \frac{\pi}{4}\int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\,d\phi$$

$$\boxed{I = \frac{\pi^2}{8}}$$

Scanned by CamScanner

Type III - Evaluation of Double integral by changing the order of Integration.

If Given integration $\int_{x=a}^{x=b} \left[\int_{y=f_1(x)}^{y=f_2(x)} F(x,y)\,dy \right] dx$ ---①

if it is difficult or impossible to integrate w.r.t. y then we change order of integration.

$$\int_{y=c}^{y=d} \left[\int_{x=\phi_1(y)}^{x=\phi_2(y)} F(x,y)\,dx \right] dy \quad ---(2)$$

Q.1 Evaluate $\int_{y=0}^{1} \int_{x=0}^{\sqrt{1-y^2}} \frac{\cos^{-1}x\,dx\,dy}{\sqrt{1-x^2-y^2}\,\sqrt{1-x^2}}$ — ①

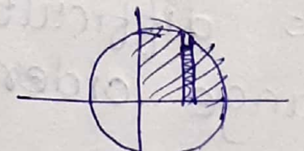
The integrand in eqⁿ ① is complicated to integrate w.r.t x but easy to integrate w.r.t y. ∴ It is required to change the order of integration.

The limits are $x=0$, $x=\sqrt{1-y^2}$ or
$x^2 = 1-y^2$
$y=0$, $y=1$ $\quad x^2+y^2=1$

use verticle strip.

$y=0$ to $y=\sqrt{1-x^2}$ &

$x=0$ to $x=1$



$$I = \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} \frac{\cos^{-1}x\,dy}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}} \right) dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}} \int_{y=0}^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}}$$

$$I_1 = \int_0^{\sqrt{1-x^2}} \frac{dy}{\sqrt{1-x^2-y^2}} = \int_0^{m} \frac{dy}{\sqrt{m^2-y^2}} \quad \text{put } \sqrt{1-x^2} = m$$

$$= \sin^{-1}\frac{y}{m}\Big]_0^m$$

$$= \sin^{-1}(1) - \sin^{-1}(0)$$

$$I_1 = \pi/2$$

$$\therefore I = \frac{\pi}{2}\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}\,dx$$

put $\cos^{-1}x = u$

$\frac{-1}{\sqrt{1-x^2}}\,dx = du$

x	0	1
u	$\pi/2$	0

$$= \frac{\pi}{2}\int_{\pi/2}^{0} -u\,du$$

$$= \frac{\pi}{2}\int_0^{\pi/2} u\,du$$

$$= \frac{\pi}{2}\left[\frac{u^2}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2}\right)^2 = \frac{\pi^3}{16}$$

$$\therefore I = \frac{\pi^3}{16}$$

Q.2 Evaluate $\int_0^a \left[\int_{y^2/a}^{y} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}\right]$

It is difficult to integrate w.r.t. x first.

$\therefore$ change order of integration.

$\therefore$ Find the limits of y use vertical strip.

for curve $x = y^2/a$, $x = y$. & $y = 0$, $y = a$

$\Rightarrow y^2 = ax$ put $y = x$

$x^2 = ax$

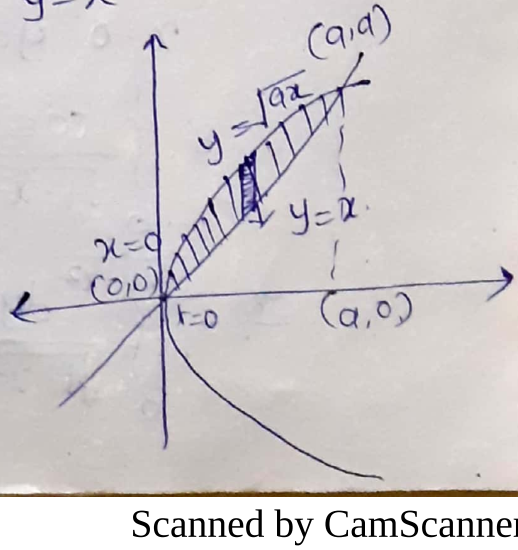
$\Rightarrow x(x-a) = 0$

$\Rightarrow x = 0$, $x = a$

$\Downarrow$ $\Downarrow$

$y = 0$ $y = 0$

$O(0,0)$, $A(a,0)$.



Y varies from $y=x$ to $y=\sqrt{ax}$ &
moving strip from $x=0 \to x=a$

$$\therefore I = \int_0^a \frac{1}{a-x}\left(\int_x^{\sqrt{ax}} \frac{y\,dy}{\sqrt{ax-y^2}}\right)dx \quad \text{——— (2)}$$

$$I_1 = -\frac{1}{2}\int_x^{\sqrt{ax}} \frac{-2y\,dy}{\sqrt{ax-y^2}}$$

put $y^2 = t$
$\Rightarrow 2y\,dy = dt$

y	x	$\sqrt{ax}$
t	x^2	ax

$$\int_{x^2}^{ax} \frac{dt}{\sqrt{ax-t}} = \int_{x^2}^{ax} (ax-t)^{-1/2}\,dt$$

$$= \left[-\frac{(ax-t)^{1/2}}{1/2}\right]_x$$

$$= -\frac{1}{2}\int_x^{\sqrt{ax}} \left[2\sqrt{ax-y^2}\right]_x^{\sqrt{ax}}$$

$$I_1 = \sqrt{x}\,\sqrt{a-x}$$

$\therefore$ from eqn (2),

$$I = \int_0^a \frac{\sqrt{x}\,\sqrt{a-x}}{a-x}\,dx$$

$$= \int_0^a x^{1/2}(a-x)^{-1/2}\,dx$$

put $x = at$, $dx = a\,dt$

x	0	a
t	0	1

$$\therefore I = \int_0^1 a^{1/2}\,t^{1/2}\,(a^{-1/2})(1-t)^{-1/2}\,a\,dt$$

$$I = \pi a/4$$

Q. 4 Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\,dx\,dy$

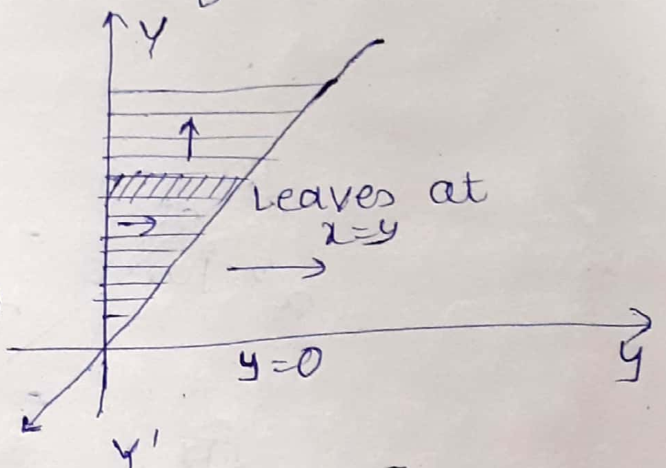
Let $I = \int_0^\infty \left[\int_{y=x}^{y=\infty} \frac{e^{-y}}{y}\,dy \right] dx$ ---- (1)

integral is difficult to integrate w.r.t. y first but easy to integrate w.r.t. x.

$\therefore$ change the order of integration.

$y = x$, $y = \infty$

$x = 0$, $x = \infty$



The new limits draw a horizontal strip. (||el to x-axis) over x' which x-varies from $x=0 \to x=y$

& moving the strip vertically from $y=0$ to $y=\infty$

from (1)

$$I = \int_{y=0}^{\infty} \left(\int_{x=0}^{x=y} \frac{e^{-y}}{y}\,dx \right) dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y} \int_{x=0}^{y} 1\,dx \right) dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y\,dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} [y]\,dy$$

$$= \int_0^\infty e^{-y}\,dy = \left. \frac{e^{-y}}{-1} \right]_0^\infty$$

$$= -1 \left[e^{-\infty} - e^{0} \right]$$

$$= -1(-1) = 1 \qquad \boxed{I = 1}$$

Scanned by CamScanner

Type II :- Transformation of Double integral into polar form or co-ordinates.

To transform the double integral

$$I = \iint_R f(x,y)\, dx\, dy \quad \text{--- (1)}$$

into polar co-ordinates (usually if the integrand is of the form $\frac{x^2y^2}{x^2+y^2}$, $(x^2+y^2)^{n/2}$, $\log(x^2+y^2)$ or $\sin(x^2+y^2)$ etc. & the region of integration is circular or elliptical boundaries, $x^2+y^2=a^2$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(x-a)^2+y^2=a^2$ etc.

by putting $x = r\cos\theta$, $y = r\sin\theta$

$\therefore x^2+y^2=r^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

& $dx\,dy = |J|\, d\theta\, dr$

$$= \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| d\theta\, dr.$$

$$= r\, d\theta\, dr.$$

$\therefore$ From equation (1), we have.

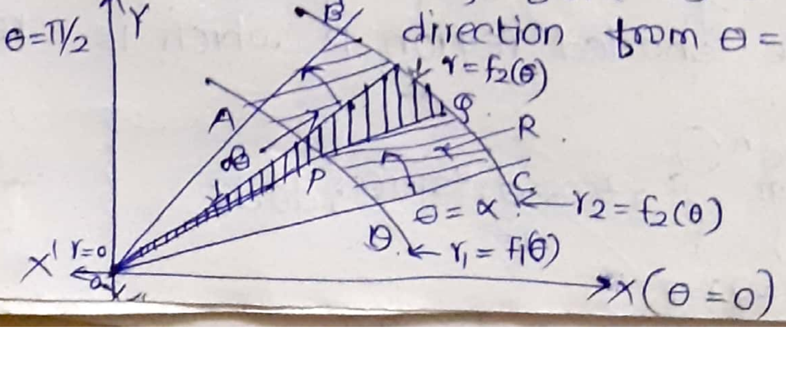
$$I = \iint_R f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr.$$

$$= \iint_R F(r,\theta)\, r\, d\theta\, dr. \quad \text{--- (2)}$$

To find corresponding limits for r, θ.

Draw a radial strip OPQ in the region of integration R from the pole $(r=0)$ over which 'r' varies from $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (for region R)

Rotate the strip OPQ from CD to AB (In anticlockwise direction from $\theta = \alpha$ to $\theta = \beta$.



∴ from eqn (2)

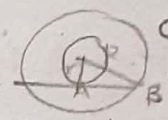
$$I = \int_{\theta=\alpha}^{\theta=\beta} \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} F(r,\theta)\, r\, d\theta\, dr \quad \text{——— (3)}$$

It is clear that the integrand to be integrate w.r.t. r first over the limits $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ (Keeping θ constant)

$$I = \int_{\theta=\alpha}^{\theta=\beta} \left(\int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} F(r,\theta)\, dr \right) d\theta \quad \text{——— (4)}$$

Example :- Evaluate

$\iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$, where R is annulus betn $x^2+y^2=4$ & $x^2+y^2=9$.



Ring shaped object, a region bounded by two concentric circles, consisting the area betn

Soln :- Let $I = \iint_R \frac{x^2y^2}{x^2+y^2}\,dx\,dy$ ——— ①

The integrand is difficult to integrate w.r.t. x or w.r.t. y and it is useful if we transform into polar co-ordinates.

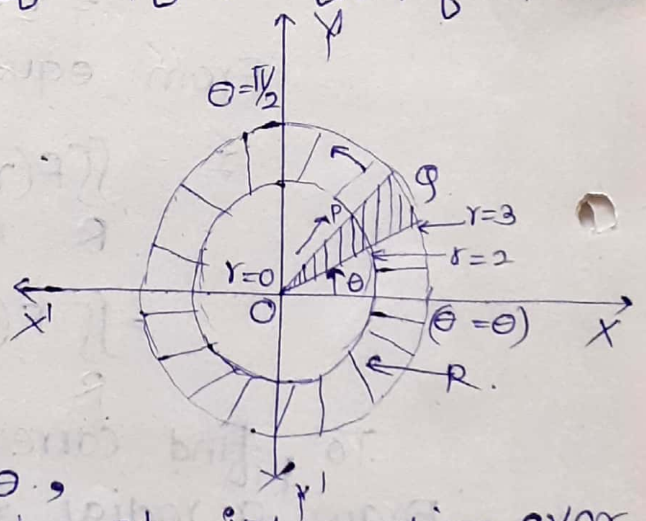
∴ put $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,d\theta\,dr$

& $x^2+y^2 = r^2$

$x^2+y^2 = 4 \Rightarrow r = 2$

$x^2+y^2 = 9 \Rightarrow r = 3$



To find limits for r & θ,
draw a strip OPQ in the region of integration over which 'r' varies from r=2 to r=3.

Now rotating the strip from θ=0 to θ=2π we get entire shaded region R which is region of integration.

∴ From eqn ①

$$I = \int_{\theta=0}^{2\pi} \int_{r=2}^{3} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2}\, r\, d\theta\, dr$$

$$= \int_{0}^{2\pi}\int_{r=2}^{3} r^3 \sin^2\theta \cos^2\theta \, d\theta \, dr.$$

$$= \int_{0}^{2\pi} \sin^2\theta \cos^2\theta \left[\int_{r=2}^{3} r^3 \, dr\right] d\theta.$$

$$= \int_{0}^{2\pi} \sin^2\theta \cos^2\theta \left(\frac{r^4}{4}\right)_2^3 d\theta.$$

$$= 4\int_{0}^{\pi/2} \sin^2\theta \cos^2\theta \left[\frac{(3)^4}{4} - \frac{(2)^4}{4}\right] d\theta.$$

$$= 65 \int_{0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta.$$

$$I = 65\left[\frac{(2-1)(2-1)}{4 \cdot 2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{65\pi}{16}.$$

H.W

Q:- Evaluate $\displaystyle\int_{0}^{a/\sqrt{2}}\int_{x}^{\sqrt{a^2-x^2}} \frac{x \, dx \, dy}{\sqrt{x^2+y^2}}.$

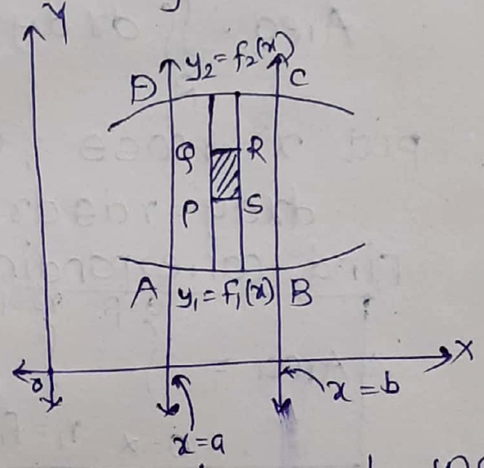
Soln:-

Area, Mass, Volume, mean and R.M.S. Values, centre of Gravity & Moment of Inertia.

Area :-

(a) Area in cartesian co-ordinate system :-

Let R be the area enclosed by the curves $y_1 = f_1(x)$, $y_2 = f_2(x)$, $x = a$ & $x = b$ as



Area of rectangle PQRS
$= \delta x\, \delta y$

Area of vertical strip

$$= \lim_{\delta x \to 0} \sum_{y_1}^{y_2} \delta x\, \delta y = \delta x \int_{y_1}^{y_2} dy$$

Adding all such strips from $x = a$ to $x = b$ we get

$$\text{Area ABCD} = \lim_{\delta x \to 0} \sum_{a}^{b} \delta x \int_{y_1}^{y_2} dy$$

$$= \int_a^b dx \int_{y_1}^{y_2} dy$$

$$\text{or } \iint_R dx\,dy = \int_{x=a}^{x=b} \int_{y_1 = f_1(x)}^{y_2 = f_2(x)} dx\,dy \quad \text{——— (1)}$$

similarly, if x having variable limits i.e. $x = \phi_1(y)$, & $x = \phi_2(y)$ then.

$$\text{Area} = \int_{y=c}^{y=d} \int_{x_1 = \phi_1(y)}^{x_2 = \phi_2(y)} dy\, dx \quad \text{——— (2)}$$

To be integrated w.r.t. x first over the limits $x_1 = \phi_1(y)$ to $x_2 = \phi_2(y)$.

Note:- (1) The area bounded by the curve $y = f(x)$, the x-axis & the lines $x = a$, $x = b$ is given by,

$$\text{Area} = \int_{x=a}^{b} y\, dx = \int_a^b f(x)\, dx.$$

(2) The area bounded by the curve $x=\phi(y)$, y-axis and the lines $y=c$, $y=d$ is given

$$\text{Area} = \int_{y=c}^{d} x\,dy = \int_{c}^{d} f(y)\,dy.$$

Ⓑ Area in polar form:-

$$\text{Area} = \iint_R dx\,dy \quad \text{.... (cartesian form)} — (1)$$

put $x = r\cos\theta$, $y = r\sin\theta$,

$dx\,dy = r\,d\theta\,dr$.

Find corresponding limits for r and θ.

$$\therefore \boxed{\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \int_{r_1=f_1(\theta)}^{r_2=f_2(\theta)} r\,d\theta\,dr} \quad \text{..... (polar form)}$$

to be evaluated w.r.t. r first over the limits $r=f_1(\theta)$ to $r=f_2(\theta)$ and then w.r.t. θ over the limits $\theta=\alpha$ to $\theta=\beta$.

Example:-

(1) cartesian form:-

① Find area bounded by the curves $y^2=4ax$ and $x^2=4ay$.

Soln:-

$$\text{Area} = \iint_A dx\,dy \quad —①$$

where $A\,(y^2=4ax,\ x^2=4ay)$

At A, $y^2=4ax$, $\dfrac{x^4}{16a^2} = y^2 = 4ax$.

$$\frac{x^4}{16a^2} - 4ax = 0$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

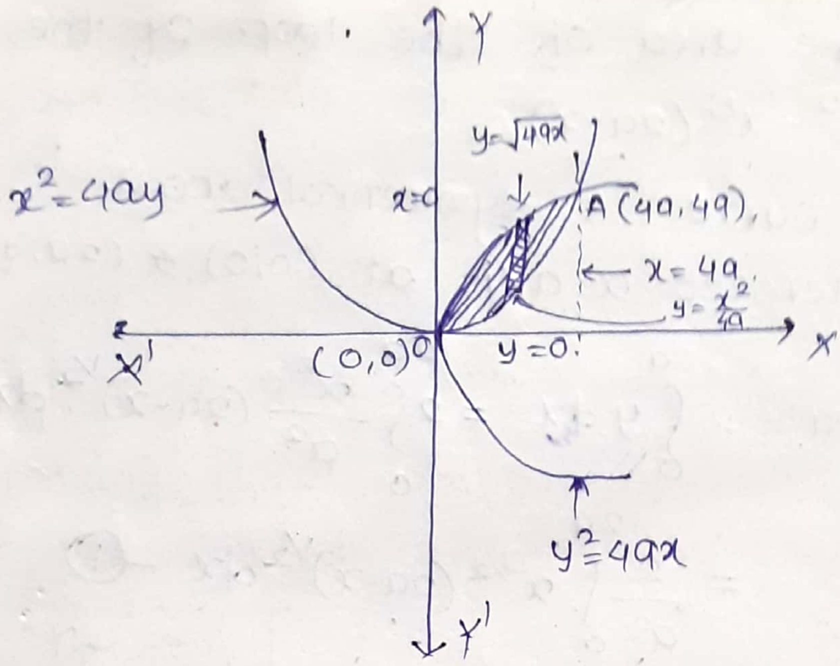
$$\Rightarrow x=0, \quad x^3 = 64a^3$$

$$\Rightarrow x=0, \quad x=4a.$$

$\therefore$ O(0,0) and A(4a, 4a) are the point of intersectio of two curves.

Scanned by CamScanner

5)



In the shaded area, over the strip y varies from $y=\frac{x^2}{4a}$ to $y=\sqrt{4ax}$.

& x varies from

$x=0$ to $x=4a$ — from ①

$$\therefore \text{Area} = \int_{x=0}^{4a}\int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx\,dy$$

$$= \int_0^{4a} (y)_{x^2/4a}^{\sqrt{4ax}}\, dx$$

$$= \int_0^{4a} \left(\sqrt{4ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[\sqrt{4a}\,\frac{x^{3/2}}{3/2} - \frac{1}{4a}\,\frac{x^3}{3}\right]_0^{4a}$$

$$= \left[\sqrt{4a}\,(4a)^{3/2}\cdot\frac{2}{3} - \frac{1}{4a\times 3}(4a)^3\right]$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\boxed{\text{Area} = \frac{16a^2}{3}}$$

Scanned by CamScanner

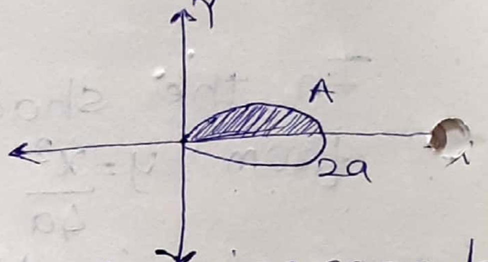
where

Que:- Find the area of the loop of the curve when $a^4y^2 = x^5(2a-x)$.

Solⁿ:- The curve is symmetrical about X-axis and intersect x-axis at (0,0) & (2a,0) we have,

$$\text{Area} = 2\int_0^{2a} y\,dx = 2\int_0^{2a} \frac{x^{5/2}}{a^2}(2a-x)^{1/2}\,dx$$

$$= \frac{2}{a^2}\int_0^{2a} x^{5/2}(2a-x)^{1/2}\,dx \quad \text{--- (1)}$$



put $x = 2a\sin^2\theta$.

$dx = 4a\sin\theta\cos\theta\,d\theta$.

$$\text{Area} = \frac{2}{a^2}\int_0^{\pi/2} (2a)^{5/2}\sin^5\theta\,(2a)^{1/2}\cos\theta\,(4a)\sin\theta\cos\theta\,d\theta$$

$$= 64a^2\int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta$$

$$= 64a^2\left[\frac{5\cdot3\cdot1\,(1)}{8\cdot6\cdot4\cdot2}\right]\frac{\pi}{2}$$

$$= \frac{5}{4}\pi a^2.$$

Que:- Find area between the curves $y^2 = 4x$ & $2x - 3y + 4 = 0$

Solⁿ:- $\text{Area} = \iint_A dx\,dy$ --- (1)

where A is Area betⁿ $y^2 = 4x$ & $2x - 3y + 4 = 0$

To find the pt. of intersection B & C.

$y^2 = 4x, \quad 2x - 3y + 4 = 0$

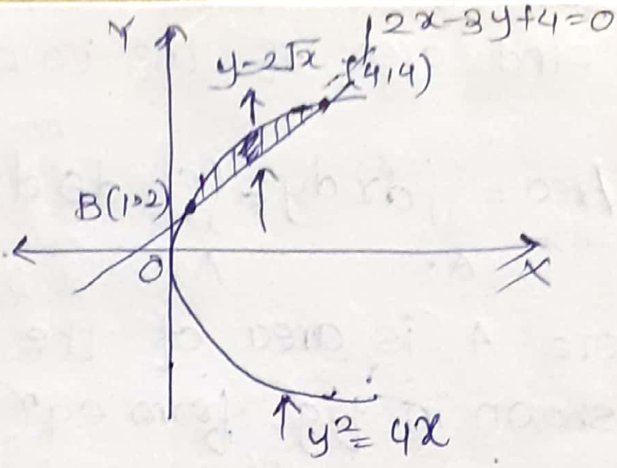
$y^2 = 2(2x)$.

$= 2(-4 + 3y)$

$\Rightarrow y^2 - 6y + 8 = 0 \quad \Rightarrow y = 2,\ y = 4$

when $y=2$, $x=1$
$y=4$ $x=4$

$\therefore B=(1,2)$, $C=(4,4)$.



limits over the strip

$y=\frac{2x+4}{3}$ to $y=2\sqrt{x}$

and moving the strip from $x=1$ to $x=4$ we get shaded area required.

$$\text{Area}=\int_{x=1}^{4}\left[\int_{y=\frac{2x+4}{3}}^{y=2\sqrt{x}} dy\right]dx$$

$$=\int_1^4\left[2\sqrt{x}-\left(\frac{2x+4}{3}\right)\right]dx.$$

$$=\left[2\,\frac{2}{3}x^{3/2}-\frac{x^2+4x}{3}\right]_1^4$$

$$=\left[\frac{32}{3}-\frac{32}{3}\right]-\left[\frac{4}{5}-\frac{5}{3}\right]$$

$$\boxed{\text{Area}=\frac{1}{3}}$$

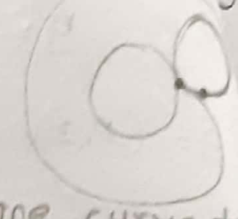
(II) Area in polar - co-ordinates :-

<u>EX:</u> Find the area common to the circles $x^2+y^2=9$ and $x^2+y^2=6x$.

Sol^n:- Given circles are $x^2+y^2=9$ and $x^2+y^2-6x=0$
Equation of both circles in polar form is $r=3$
&

Ex:- Find area of the cardiode $r = a(1+\cos\theta)$

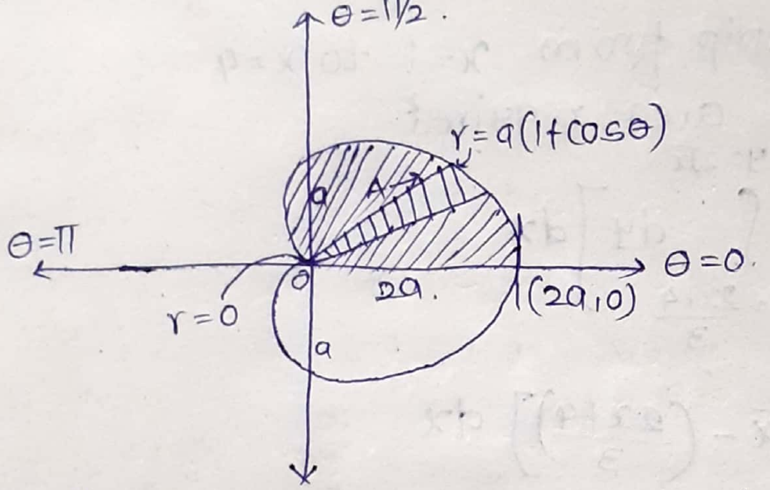
ass:-



Soln:- $\text{Area} = \iint_A dx\,dy = \iint_A r\,d\theta\,dr$ — ①

where A is area of the cardiode as shown in fig, from eqn ①.

plane curve trac[ed] by a pt. on the perimeter of a c[ircle] that is rolling arou[nd] a fixed circle of the same radius



Area = 2x Area of upper half

$$= 2\iint_A r\,d\theta\,dr = 2\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} r\,d\theta\,dr.$$

$$= 2\int_{\theta=0}^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{\pi}(1+2\cos\theta+\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left[1+2\cos\theta+\frac{1+\cos 2\theta}{2}\right]d\theta$$

$$= a^2\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= a^2\left[\frac{3\pi}{2}+0\right]$$

$$\boxed{\text{Area} = \frac{a^2 3\pi}{2}}$$

Scanned by CamScanner

Mass:- $M = \int \rho \, dA$

$\Rightarrow M = \text{Mass} = \iint_R \rho \, dx \, dy = \iint_R f(x,y) \, dx \, dy$ —— (1)

similarly, In polar co-ordinates

if $\rho = F(r,\theta)$

then $\text{Mass} = \iint_R F(r,\theta) \, r \, d\theta \, dr$ —— (2)

Volume of solids:-

(1) The volume of solid by triple integration is given by

$$\text{Volume} = V = \iiint_V dv = \iiint_V dx\,dy\,dz \quad \text{—— (1)}$$

(2) In spherical polar co-ordinates

$$V = \iiint_V r^2 \sin\theta \, dr \, d\theta \, d\phi \quad \text{- - (2)}$$

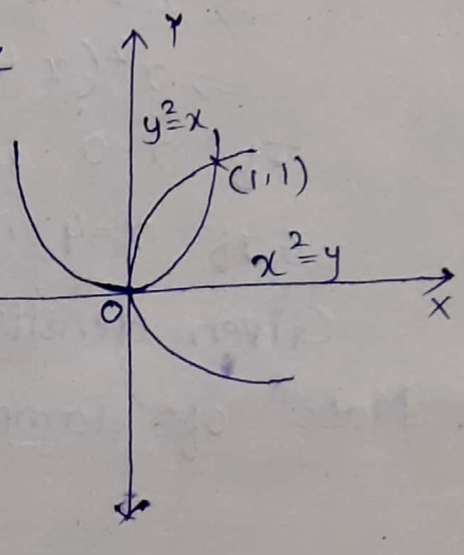
(3) In cylindrical polar co-ordinates.

$$V = \iiint_V \rho \, d\rho \, d\phi \, dz \quad \text{—— (3)}$$

Que:- Find the volume bounded by the cylinders $y^2 = x$, $x^2 = y$ and planes $z = 0$ and $x + y + z = 2$.

Soln:-

$$\text{volume}(V) = \iiint dx\,dy\,dz = \iint dx\,dy \int_0^{2-x-y} dz$$



$$V = \iint_R (2-x-y) \, dx \, dy$$

where R is the region in XOY plane as shown in fig.

$$V = \int_0^1 \int_{x^2}^{\sqrt{x}} (2-x-y) \, dy \, dx.$$

$$= \int_0^1 \left[2y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx$$

$$V=\int_0^1\left[\left(2\sqrt{x}-x^{3/2}-\frac{x}{2}\right)-\left(2x^2-x^3-\frac{x^4}{2}\right)\right]dx$$

$$V=\left[\frac{2x^{3/2}}{3/2}-\frac{x^{5/2}}{5/2}-\frac{x^2}{4}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{10}\right]_0^1$$

$$V=\frac{4}{3}-\frac{2}{5}-\frac{1}{4}-\frac{2}{3}+\frac{1}{4}+\frac{1}{10}$$

$$V=\frac{11}{30}$$

Mass:-

Ex:- Find the mass of lamming bdd by the curve $16y^2=x^3$ & the line $2y=x$ assuming that the density at a pt. of the area varies as the distance of the pt from x-axis:-

Soln:- Given curves are

$16y^2=x^3$ & $2y=x$ —(1)

To find the pt of intersection of the curves are have,

$$16\left(\frac{x}{2}\right)^2=x^3$$

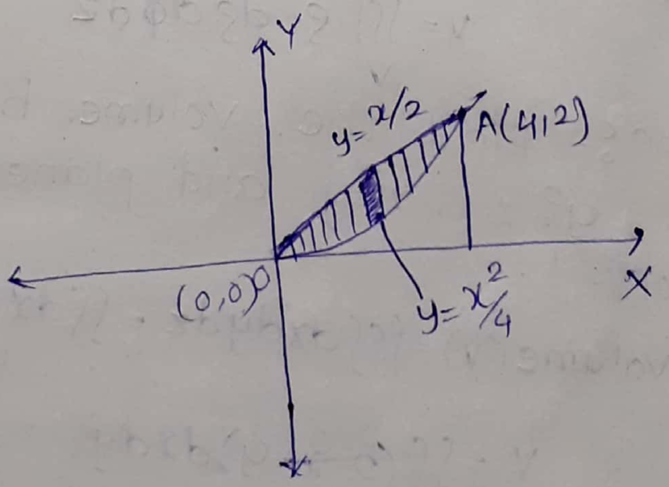
$$4x^2=x^3$$

$$\Rightarrow x^3-4x^2=0$$

$$\Rightarrow x^2(x-4)$$

If $x=0$, $y=0$

& If $x=4$, $y=2$.



Given density of any pt (x,y) is ky

Mass of lamming $=\iint_A \rho\, dx\,dy$

$$=\iint_A ky\,dx\,dy$$

$$=k\int_0^4\left(\int_{y=x^{3/2}/4}^{x/2} y\,dy\right)dx$$

$$= k\int_0^4 \left[\frac{y^2}{2}\right]_{\frac{x^{3/2}}{4}}^{x/2} dx$$

$$= \frac{k}{2}\int_0^4 \left(\frac{x^2}{4} - \frac{x^3}{16}\right) dx$$

$$= \frac{k}{2}\left(\frac{x^3}{12} - \frac{x^4}{64}\right)_0^4$$

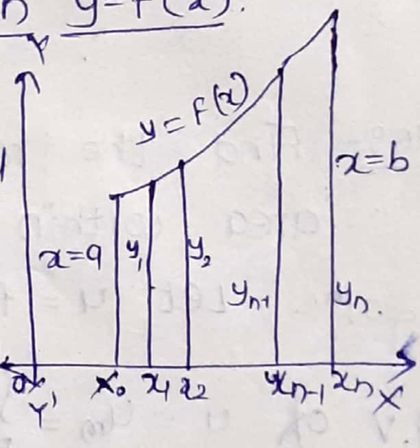
$$= \frac{k}{2}\left(\frac{4^3}{12} - 4\right)$$

$$\boxed{M = \frac{2}{3}k}$$

Mean & R.M.S. value :-

① Mean value of the function $y = F(x)$.

The mean value of the function $y = f(x)$ over the interval $x = a$ to $x = b$ is given.



$$y_m = \frac{\int_a^b f(x)\,dx}{b-a} = \frac{1}{b-a}\int_a^b F(x)\,dx \quad \text{—①}$$

lly, mean square value of the function $y = f(x)$ over $x = a$ to $x = b$ is.

$$y_{m.s.} = \frac{\int_a^b (F(x))^2 dx}{b-a} = \frac{1}{a-b}\int_a^b (F(x))^2\,dx \quad \text{—(2)}$$

② The mean value of the function $z = f(x,y)$ over the region R is given by

$$z_m = \frac{\iint_R f(x,y)\,dx\,dy}{\iint_R dx\,dy} \quad \text{—(3)}$$

(3) The mean value of function.

$u = F(x,y,z)$ over the volume V is given by

$$u_m = \frac{\iiint_V F(x,y,z)\,dx\,dy\,dz}{\iiint_V dx\,dy\,dz} \quad \text{——— (4)}$$

(4) <u>Root Mean square value</u> (R.M.S value):-

- generally used in A.C. electrical circuits & related to periodic functions.
- If $y = F(x)$ with period P. then R.M.S. value of y is given by,

$$y_{r.m.s} = \sqrt{\frac{\int_c^{c+p} y^2\,dx}{\int_c^{c+p} dx}} \quad (c \text{ is constant})$$

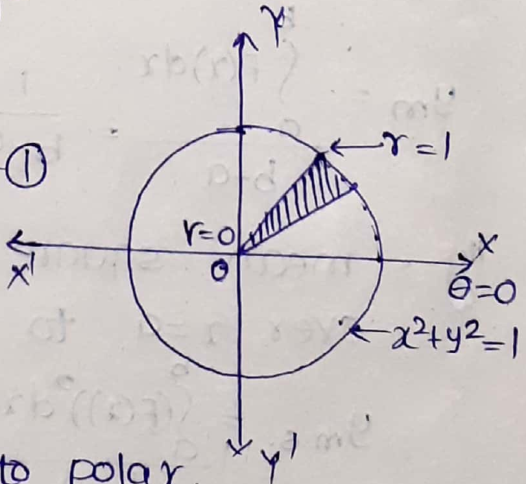
$$= \sqrt{\frac{\int_c^{c+p} [F(x)]^2\,dx}{\int_c^{c+p} dx}} \quad \text{——— (5)}$$

<u>Que:-</u> Find the mean value of $e^{-(x^2+y^2)}$ over the area within the circle $x^2+y^2=1$.

Soln:- Let $u = f(x,y) = e^{-x^2-y^2}$

$$\text{M.V of } u = u_m = \frac{\iint_R e^{-x^2-y^2}\,dx\,dy}{\iint_R dx\,dy} \quad \text{——— (1)}$$

where R is circle, $x^2+y^2=1$



Transforming integral (1) into polar.

$r=0 \longrightarrow r=1$

$\theta=0 \longrightarrow \theta=2\pi$

$$\therefore u_m = \frac{\iint_{\theta=0,\,r=0}^{2\pi,\,1} e^{-r^2}\,r\,d\theta\,dr}{\text{Area of circle } x^2+y^2=1}$$

Scanned by CamScanner

$$= \frac{\int_0^{2\pi} -\frac{1}{2}\left[\int_0^1 e^{-r^2}(-2r)\,dr\right] d\theta}{\pi(1)^2} \qquad \left(\because \int e^{f(x)} f'(x)\,dx = e^{f(x)}\right)$$

$$= \frac{1}{\pi}\left(\frac{-1}{2}\right)\int_0^{2\pi}\left(e^{-r^2}\right)_0^1 d\theta$$

$$= \frac{-1}{2\pi}\int_0^{2\pi}\left(e^{-1}-1\right) d\theta$$

$$= \frac{-1}{2\pi}\left(\frac{1}{e}-1\right)\left[\theta\right]_0^{2\pi}$$

$$\boxed{u_m = 1-\frac{1}{e}}$$

Centre of Gravity and Moment of Inertia:-

Centre of Gravity :- defn:-

Centre of Gravity of a body is defined as the pt through which resultant weight of the body acts.

If $m_1, m_2, m_3 \dots m_n$ are pt of masses situated at the points (x_1, y_1, z_1) (x_2, y_2, z_2) $\dots$ (x_n, y_n, z_n) & $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of C.G of system then.

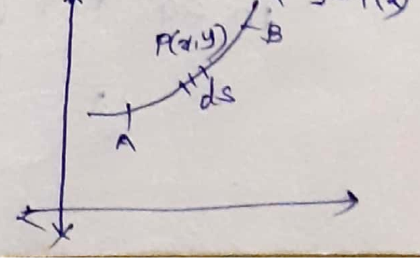
$$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm}, \quad \bar{z} = \frac{\int z\,dm}{\int dm}.$$

(A) C.G of an Arc:-

Let the mass be distributed in the form of curve $y = f(x)$, 'ds' be an elementary arc at the point $P(x, y)$.

If ρ is density at the point $P(x,y)$ then mass of this element is, $dm = \rho\,ds$ —①

$$\bar{x} = \frac{\int x\,ds}{\int ds}, \quad \bar{y} = \frac{\int y\,ds}{ds}.$$



Note:- (1) If $y=f(x)$, then $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$.

(2) If $x = f(y)$, then $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

(3) If $x = f_1(t)$, $y = f_2(t)$ then.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,.dt$$

(4) If $r = f(\theta)$ then $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,.d\theta$.

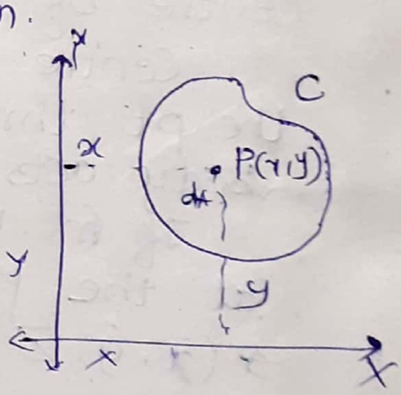
(5) If $\theta = f(r)$ then $ds = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\, dr$

(B) C.G of plane lammina:-

Let $(\bar{x}, \bar{y})$ be the co-ordinates of C.G of plane lammina bounded by the curve c & ρ is density at the point $P(x,y)$ then.

$$dm = \rho\, dA \quad \text{—①}.$$

$$\bar{x} = \frac{\iint_R x\rho\, dx\, dy}{\iint_R \rho\, dx\, dy}, \quad \bar{y} = \frac{\iint_R y\rho\, dx\, dy}{\iint_R \rho\, dx\, dy} \quad \text{—}(*)$$



If ρ is const. then.

$$\bar{x} = \frac{\iint_R x\, dx\, dy}{\iint_R dx\, dy}, \quad \bar{y} = \frac{\iint_R y\, dx\, dy}{\iint_R dx\, dy} \quad \text{—}(*)(*)$$

where R is region bounded by the curve C or Lamina.

Note:- For the polar curve, put $dA = r\, d\theta\, dr$, $x = r\cos\theta$, $y = r\sin\theta$. in eq^n (*) + (*)(*), we get.

$$\bar{x} = \frac{\iint_R r\cos\theta\, \rho\, r\, d\theta\, dr}{\iint_R \rho\, r\, d\theta\, dr}.$$

$$\bar{y} = \frac{\iint_R r\sin\theta \, \rho \, r \, d\theta \, dr}{\iint_R \rho \, r \, d\theta \, dr}$$

(C) C.G of a solid:-

If the $(\bar{x}, \bar{y}, \bar{z})$ be co-ordinates of C.G. of the solid which encloses volume V.

the solid which encloses volume V then. ρ is density at the point $P(x, y, z)$ then.

$$dm = \rho \, dv = \rho \, dx \, dy \, dz \quad \text{---(1)}$$

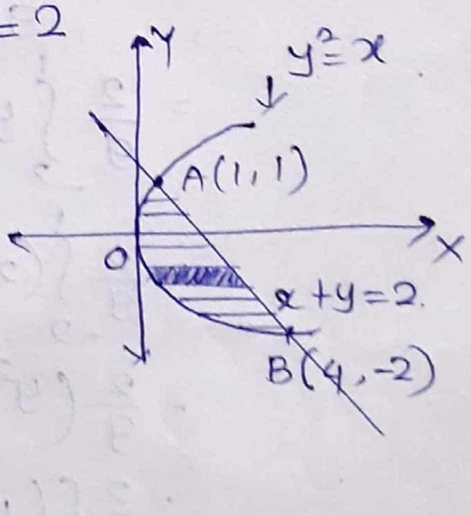
$$\therefore \bar{x} = \frac{\iiint_V x \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}, \quad \bar{y} = \frac{\iiint_V y \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}$$

$$\& \ \bar{z} = \frac{\iiint_V z \rho \, dx \, dy \, dz}{\iiint_V \rho \, dx \, dy \, dz}.$$

EX:- (2) on C.G of Area

Find the C.G. of the area bounded by $y^2 = x$ & $x + y = 2$.

Soln:- Let $(\bar{x}, \bar{y})$ be co-ordinates of C.G. of area bounded by $y^2 = x$ & $x + y = 2$



$$\bar{x} = \frac{\iint x \, dx \, dy}{\iint dx \, dy} = \frac{N}{D}$$

$$\text{where } N = \int_{-2}^{1} \int_{y^2}^{2-y} x \, dx \, dy$$

$$= \int_{-2}^{1} \left(\frac{x^2}{2}\right)_{y^2}^{2-y} dy$$

$$= \frac{1}{2} \int_{-2}^{1} \left[(2-y)^2 - y^4\right] dy$$

$$= \frac{1}{2}\left[\frac{(2-y)^3}{-3} - \frac{y^5}{5}\right]_{-2}^{1}$$

$$= \frac{1}{2}\left[\left(\frac{13}{3} - \frac{11}{5}\right) - \left(\frac{-56}{3} + \frac{32}{5}\right)\right]$$

$$= \frac{36}{5}$$

& $A = \iint dx\,dy$

$$= \int_{-2}^{1}\int_{x=y^2}^{2-y} dx\,dy$$

$$= \int_{-2}^{1} (2-y-y^2)\,dy$$

$$= \left(2y - \frac{y^2}{2} - \frac{y^3}{3}\right)_{-2}^{1}$$

$$= 9/2$$

Now, $\bar{y} = \dfrac{\iiint y\,dx\,dy}{\iint dx\,dy}$

$$= \frac{\int_{-2}^{1}\int_{y^2}^{2-y} y\,dx\,dy}{9/2}$$

$$= \frac{2}{9}\int_{-2}^{1} y\left[(2-y)-y^2\right]dy$$

$$= \frac{2}{9}\int_{-2}^{1}(2y - y^2 - y^3)\,dy$$

$$= \frac{2}{9}\left(y^2 - \frac{y^3}{3} - \frac{y^4}{4}\right)_{-2}^{1}$$

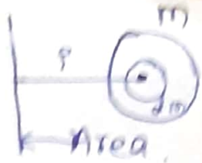
$$= \frac{2}{9}\left[\left(1 - \frac{1}{3} - \frac{1}{4}\right) - \left(4 + \frac{8}{3} - \frac{16}{4}\right)\right]$$

$$= \frac{2}{9}\left(-\frac{27}{12}\right) = \frac{2}{9}\left(-\frac{9}{4}\right)$$

$\bar{y} = -1/2$

$$\boxed{\bar{x} = \frac{8}{5}, \quad \bar{y} = \frac{-1}{2}}$$

Moment of Inertia:-



(A) $M.I = \lim_{dm \to 0} \sum p^2 dm = \int p^2 dm.$

(B) M.I of a plane lamina:-

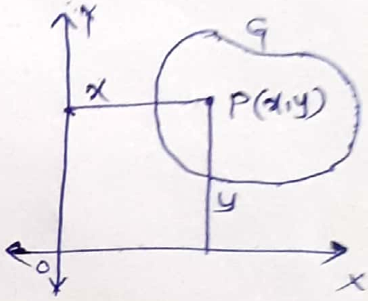
Consider a plane lamina R bounded by the curve c.

If ρ is density at the point P(x, y) then

$dm = \rho\, dx dy$

(i) If p is the distance of this elementary mass from the axis, the M.I about this axis is.

$$M.I = \iint_R \rho\, p^2 dx\, dy.$$



(ii) The M.I of the lamina about x-axis is.

$$M.I = \iint_R \rho\, y^2 dx\, dy.$$

(iii) The M.I of lamina about Y axis is

$$M.I = \iint_R \rho\, x^2 dx dy \qquad (\because p = x)$$

(iv) The M.I in polar co-ordinates is

$$M.I = \iint_R \rho\, p^2\, r dr d\theta.$$

~~(C) M.I of solid~~ :-